Research

Connected Decycling Number of \(P_m \times P_n\)*

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**Abstract:** There are many results on the decycling number \(\chi(G)\) of a graph \(G\) is the smallest number of vertices can be removed from a graph \(G\) so that the resultant graph contains no cycle, that is, induced forest. A decycling set containing exactly \(\chi(G)\) vertices of \(G\) is called \(\chi\) set. There are many graph theorist they paid attention to this topic, while few graph theorist paid their attention to find the maximum induced tree of \(G\) and its important applications. In this paper, we are motivated to compute the connected decycling number of Cartesian product of two paths \(P_m \times P_n\). So, it may be define as, the maximum independent set of vertices of a graph \(G\) whose removal leaves a connected acyclic graph (or connected forest) is called a connected decycling set, and it is denoted by \(I\). The maximum independent number of a connected decycling set of \(G\) referred to as the connected decycling number of \(G\), designated by \(\tau(G)\). In this paper, we compute the connected decycling number for the family of graphs consisting of the Cartesian product of two paths, that is, \(\tau(P_m \times P_n)\), and obtained some new results. Mainly, we prove that

\[
\tau(P_m \times P_n) \geq \left\lfloor \frac{mn - m - n + 2}{3} \right\rfloor.
\]

**Keywords:** Decycling number, Induced subgraph, Tree, Grids, Graph Theory, Cycles.

Introduction.

Let \(G = (V(G), E(G))\) be a graph, where \(V(G)\) and \(E(G)\) represents the vertex set and edge set of graph respectively. In this paper, we allowed only connected simple graphs and for the standard terminologies and notations of graph theory, we simply referred to Wilson at el. (1996). Furthermore, the minimum number of edges whose removal eliminate all cycles in a given graph has been known as cycle rank and this parameter has a simple expression

\[
\beta(G) = |E(G)| - |V(G)| + c.\]

Where \(c\) is the number of components in \(G\) and on the other hand, the corresponding problem to eliminate all cycles by mean of deletion of vertices goes
back at least to the work of Kirchhoff at el. (1847) on spanning tree. In addition, this is not a simple problem it is quite difficult for some simple graphs such as planner graphs, bipartite graph etc. Suppose that \( I \) be an independent set of vertices in \( G \), and \( G - I \) is a tree, then \( I \) is said to be the connected decycling set (or CDS in short) of \( G \). The maximum size of connected decycling set of \( G \) is called the connected decycling number (or CDN in short) and it is denoted by \( \tau(G) \). A connected decycling set of this size is called \( \tau - \) set. To determine the CDN of graph is equivalent to finding the maximum order of an induce tree, the sum of two numbers equals to the order of graph \( (i.e \tau(G) + a(G) = n) \), where \( a(G) \) is the order of the largest induced tree. The connected decycling set problem has various applications in area such as combinatorial circuit design by Focardi at el. (2000), an operating system task, that is, consider the typical operating system task of allocation resources to processors while preventing deadlock this task can carried out by using vertex-removing operation on dependence graph. Suppose that a vertex represents a processor and each edge \((i,j)\) the request of processor \( i \) for a resource already allocated to processor \( j \). If the graph contains a cycle, then a deadlock has occurred and therefore processors will indefinitely wait for the requested resources. This may be solved by moving into a waiting queue the minimum number of processors so that the new dependence graph becomes deadlock free Wang at el. (1985), this technique may also be very useful for the artificial intelligence to the constraint satisfaction problem and to Bayesian inference Festa at el. (1999). Ofcouse CDN has very nice contribution in Chemical graph theory, for example to breakdown the molecular structures of compounds, and make it cycle free. Such as the enumeration problems in which molecules are regarded as being constructed from larger building blocks, than individual atoms. Those molecules constructed from benzene rings (called “benzenoid condensed polycyclic system” or “polyhexes”) provide a wealth of combinatorial problems. Again, the connected acyclic member of this set (which have been said “tree-like polyhexes”) are comparatively easy to enumerate Robald at el. (1996). Erdős at el. (1986) gives an important concept about the maximum induced tree. They explicitly investigated the relationship of maximum size of a subset of vertices of graph \( G \) that induces a tree to other parameters associated with \( G \) by Festa at el. (1999). Therefore, many graph theorist paying their attention on the decycling number of graphs (or, minimum feedback vertex) to obtain an induced forest, this problem has been well investigated ( Bau & Beinke 2002; Beinke & Vandell 1997; Caragainnis at el 2002; Festa at el 1999; Focardi at el. 2000; Madelain & Stewwart 2008 and Ren at el. 2017).

As a matter of this fact, the program of solution in this paper, is to compute the connected decycling set \( I \) of size \( \tau(P_m \times P_n) \text{ for all } mn \geq 3 \), in such a way that the resultant graph be a maximum induced tree (i.e. \( T = V(G) - I \)). Now, we have to carefully exercise on the vertex removal operation on the basses of two strict conditions. (1) All the removal vertices in \( G \) essentially independent. (2) The deletion of an independent vertices cannot isolate any vertex in \( G \). In this paper, the removal vertices and induced tree are exemplified in figures by black dots “•” and red bold bars respectively. This paper is broken up into two three sections. The first of which served as basic terminologies and notations of graph theory, the second part demonstrate the main result and discussion and third part present conclusion of paper.
Results and Discussion

As promised in the introduction we compute the connected decycling number of $P_m \times P_n$, for $m \leq 7$. The classical labeling for the grids of path $P_m$ and $P_n$ is the graph $P_m \times P_n$ with vertex set and edge set respectively

$$ V(P_m \times P_n) = \{x_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n\} $$

$$ E(P_m \times P_n) = \{x_{ij} x_{rs} \mid i = r, v_j v_s \in E(P_n) \text{ or } j = s, u_i u_r \in E(P_m)\}. $$

The margin number of connected decycling set $I$ can be define as $m(I) = c + |E(I)| - 1$, where $c$ and $|E(I)|$ are, respectively, the number of components of $G - I$ and the number of incident edges on $I$. In fact, the margin number is very useful to measure a gap between the connected decycling set in $I$ and connected decycling of $G$.

**Theorem 1.** Let $G = (P_m \times P_n)$ be a 2-dimensional grids graph with maximum degree $\Delta$ and $I$ be a connected decycling set for $G$. Suppose that $d$ ($d < \Delta$) is a fixed natural number with $I = I_\alpha \cup I_\beta$. Where, $I_\alpha = \{x \mid d_G(x) = \Delta, x \in I\}$, $I_\beta = \{x \mid d_G(x) = d < \Delta, x \in I\}$. Then

i. $(\Delta - 1)(|I| - |I_\beta|) + (d - 1)|I_\beta| = \beta(G) + m(I)$.

ii. $|I| = \frac{1}{\Delta - 1}(\beta(G) + (\Delta - d)|I_\beta| + m(I))$.

**Proof (i).** Let $I$ be a connected decycling set of $G$. Then $G - I$ has $|V(G)| - |I|$ vertices, $c$ components and $|V(G)| - |I| - c$ edges. If $I$ is removed from $G$, then, $(\Delta - 1)(|I| - |I_\beta|) + (\Delta - d)|I_\beta|$, edges are also removed. Therefore, we must have

$$ |E(G)| - (\Delta - 1)(|I| - |I_\beta|) + (d - 1)|I_\beta| = |V(G)| - |E(I)| - c. $$

$$(\Delta - 1)(|I| - |I_\beta|) + (d - 1)|I_\beta| = (|E(G)| - |V(G)|) + 1 + (c + |E(I)| - 1).$$

$$(\Delta - 1)(|I| - |I_\beta|) + (d - 1)|I_\beta| = \beta(G) + m(I).$$

**Proof (ii).** Let $I$ be a connected decycling set of $G$ and theorem 1. (i) implies that $(\Delta - 1)(|I| - |I_\beta|) - (d - 1)|I_\beta| = \beta(G) + c + E(I) - 1$.

$$(\Delta - 1)|I| - (\Delta - d)|I_\beta| = \beta(G) + c + E(|I|) - 1.$$
\[ \tau(P_m \times P_n) \geq \frac{1}{\Delta - 1} (\beta(G) + (\Delta - d)), \]

where \( d < \Delta \) represents the degree 2 or 3 vertices of \( \tau \) set of \( G \).

**Corollary 3.** If a graph \( P_m \times P_n \) contains the connected decycling set \( I \) of order \( \tau(P_m \times P_n) \).

Then, \( m(I) = 3I - mn + m + n - 2 \).

**Proof.** Theorem 1 (ii), implies that, if \( (\Delta - d)|I_{\beta}| \geq 1 \). Then \( m(I) = 3I - \beta(P_m \times P_n) - 1 \).

\[
m(I) = 3I - (E|P_m \times P_n| - V|P_m \times P_n| + 1) - 1.
\]

\[
m(I) = 3I - (m(n - 1) + n(m - 1) - mn + 1) - 1.
\]

\[
m(I) = 3I - (mn - n - m + 1) - 1.
\]

\[
m(I) = 3I - mn + m + n - 2.
\]

**Theorem 4.** If \( m, n \geq 3 \), then

\[
\left\lfloor \frac{mn - m - n + 2}{3} \right\rfloor \leq \tau(P_m \times P_n) \leq \left\lfloor \frac{mn - m - n + |I_{\beta}| + m(I) + 1}{3} \right\rfloor.
\]

**Proof.** To show, by the definition of cycle rank \( \beta(P_m \times P_n) = mn - m - n + 1 \). And corollary 2 implies that

\[
\tau(P_m \times P_n) \geq \left\lfloor \frac{\beta(P_m \times P_n) + 1}{\Delta - 1} \right\rfloor.
\]

\[
\tau(P_m \times P_n) \geq \left\lfloor \frac{mn - m - n + 2}{3} \right\rfloor.
\]

Let \( I \) be the connected decycling set of \( P_m \times P_n \), and follows theorem 1. (b), we get

\[
\tau(P_m \times P_n) \leq |I| = \left\lfloor \frac{mn - m - n + |I_{\beta}| + m(I) + 1}{3} \right\rfloor.
\]

**Lemma 5.** Let \( m, n \geq 4 \) with \( m \) even. If \( I \) is connected decycling set of \( P_m \times P_n \), then \( I(1,2) \) contains at least one vertex of degree 3 or 2 and this vertex necessarily nonadjacent with interior independent vertices of \( G \).

**Proof.** Suppose that \( m = 2k \). Then the connected subgraph induced by first two columns contains \( k \) disjoint 4-cycles, each of them have a vertex in \( I \). Moreover, an independent set of interior vertices in \( P_m \times P_n \) has at least \( k - 1 \) vertices in \( I(1,2) \), and the \( k - 1 \) vertices are definitely independent with the boundary vertex (see Fig.3 and Fig.5a).  

In order to find optimal result, we have to proof all subcases of grids graph. For this purpose, we need the following lemma
Lemma 6. Let $1 \leq q \leq n$. Then $\tau(P_m \times P_n) \geq \tau(P_m \times P_{n-q}) + \tau(P_m \times P_q)$.

Because of unique structure (i.e. degree variant), it will be better to dispose theorem 7 immediately.

Theorem 7. If $m = 2$ and $n \geq 2$ Then, $\tau(P_2 \times P_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. In $P_2 \times P_n$ there are $\left\lfloor \frac{n}{2} \right\rfloor$ disjoined 4-cycles. Therefore, every connected decycling set must contains at least $\left\lfloor \frac{n}{2} \right\rfloor$ independent vertices. In addition, it is sufficient to find to connected decycling set with $\left\lfloor \frac{n}{2} \right\rfloor$ independent vertices. Let $V(P_2) = \{1,2\}, V(P_n) = \{1,2 \ldots n\}$, and $k = \left\lfloor \frac{n}{2} \right\rfloor$.

Then $I = \{v_{1,2}, v_{1,4}, \ldots, v_{1,2k}\}$, obviously all the vertices in set $I$ are independent (see Fig.1), therefore $G - I$ is a tree.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Fig.1.}
\end{figure}

Now we are looking forward to compute the connected decycling number $\tau(P_m \times P_n)$ of some other cases.

Theorem 8. If $m = 3$ and $n \geq 3$, then $\tau(P_3 \times P_n) = \frac{3n}{4}$.

Proof. We use induction module 4. Theorem.4 implies that $\tau(P_3 \times P_3) \geq 2$, and $\tau(P_3 \times P_4) \geq 3$. Thus, the results true for $n \leq 4$. Now, we assume that $n = 4h + r, with 1 \leq r \leq 4$. Then $\tau(P_3 \times P_{4h+r}) = 3h + \left\lfloor \frac{3r}{4} \right\rfloor$, by our hypotheses;

\[
(P_3 \times P_{4(h+1)+r}) = (P_3 \times P_{4h+r}) + (P_3 \times P_4).
\]

\[
\tau(P_3 \times P_{4(h+1)+r}) \geq \tau(P_3 \times P_{4h+r}) + \tau(P_3 \times P_4) \geq 3h + \left\lfloor \frac{3r}{4} \right\rfloor + 3. \quad \text{(i.e. by lemma 6)}
\]

Such that the connected decycling set of size is $3(h + 1) + \left\lfloor \frac{3r}{4} \right\rfloor$.

The sequence of removal independent vertices also depends on the order of columns in $(P_3 \times P_n)$. For example

i. If a subgrid $(P_3 \times P_n)$ contains an odd number of columns, then the set of vertices in $I$ be as following (see Fig.2a).

\[
I_1 = \{v_{1,3}, v_{1,7} \ldots v_{1,4k-1}\}, \forall k \in N. \quad I_2 = \{v_{2,2}, v_{2,4} \ldots v_{2,2k}\}, \forall k \in N.
\]

ii. If a subgrid $(P_3 \times P_n)$ contains an even number of columns (see Fig.2b), then
\[ I_1 = \{ v_{1,4}, v_{1,8} \ldots v_{1,4k} \}, \forall k \in N. \]
\[ I_2 = \{ v_{2,1}, v_{2,3} \ldots v_{2,2k-1} \}, \forall k \in N. \]

On the basis of above observations, we may conclude that, \( I = I_1 \cup I_2 \). So, the basic conditions are satisfied, therefore \( G - I \) is a tree.

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**Theorem 9.** If \( m = 4 \) and \( n \geq 3 \), then \( \tau(P_4 \times P_n) = n \).

**Proof.** It follows from theorem 4, the \( \tau(P_4 \times P_n) \geq n \). Lemma 5, implies that \( I(1,2) \) contains a vertex of degree 3 or 2 and the rest of interior vertices in \( I \) are independent. So, the pattern of removal vertices clearly visualized in Fig.3, where exactly each column contains a single vertex. For Example, in the following Fig.3 there are three subset of vertices in \( G \), that is, \( I_1, I_2, I_3 \in I \).

\[
I_1 = \{ v_{1,1} \}, \quad I_2 = \{ v_{2,3}, v_{2,5} \ldots v_{2,2k+1} \}; \forall k \in N.
\]
\[
I_3 = \{ v_{3,2}, v_{3,4} \ldots v_{3,2k} \}; \forall k \in N. \text{ Hence, } I = I_1 \cup I_2 \cup I_3.
\]

The removal vertices in \( G \) are clearly independent and their elimination cannot isolated any sort of vertex, and evidently \( G - I \) is a connected forest.

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**Theorem 10.** If \( m = 5 \) and \( n \geq 4 \), then \( \tau(P_5 \times P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - \left\lfloor \frac{n}{8} \right\rfloor - 1 \).

This case is slightly tricky, on the other hand, our commitment be going to find its solution in an appropriate way. To insure its solution, we shall effectively use corollary 3, theorem 4, and other upcoming two lemmas (i.e. lemma 10.1 and 10.2).
Lemma 10.1. The margin of connected decycling set $I$ of order $\tau(P_5 \times P_n)$ is $m(I) = 3l - 4n + 3$.

Proof. It follows from corollary 3. ■

Lemma 10.2

(a) for $n \geq 2$, $\tau(P_5 \times P_n) = \left\lfloor \frac{4n-1}{3} \right\rfloor$.

(b) If $n = 6q$, then $\tau(P_5 \times P_n) \geq 8q$.

Proof of (a). It follows from lemma 10.1.

Proof of (b). Let $n = 6q$, and suppose that $G = P_5 \times P_n$ has a connected decycling set $I$ of order $8q - 1$. (i.e. lemma 10.2 (a)). Lemma 10.1 implies that, its margin is 0. Contrary, lemma 5 implies that there are eight an independent vertices in $G$, since two independent vertices has degree 3, one has degree 2, and the rest of five vertices has degree 4 (see Fig.4c). This is only possible when margin is not equals to 0. It means that the connected decycling set has at least 8q independent vertices. ■

If $2 \leq n \leq 9$, then lemma 10.2(a) gives the following tabular results.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau(G)$</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

Table.1.

From the Table.1 observations, we assumed two fundamental results; they may be very useful for upcoming discussion.

i. If $n \leq 8$, then $\tau(P_5 \times P_n) < \tau(P_5 \times P_{n+1})$.

ii. If $n \leq 7$, then $\tau(P_5 \times P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - 1$.

Now, we have to find CDS for $\tau(P_5 \times P_n)$ for $n \leq 9$.

Lemma 10.3. The connected decycling set of $(P_5 \times P_3)$ have a special type of independent vertices and their eliminations cannot isolate any vertex in $G$.

Proof. Let $G = P_5 \times P_3$ and suppose that $I$ be a connected decycling set of order 3. Lemma 10.1 implies that, the margin is 0, and it contains three non-adjacent vertices, in which two vertices has degree 4 and one of it has degree 3 (see Fig.4a). Thus, the removal vertices in $G$ are independent and their elimination cannot isolate any vertex in $G$. Therefore $G - I$ is a tree. ■
Lemma 10.4. The connected decycling set of \((P_5 \times P_5)\) is one of the type of independent vertices and their elimination gives a tree.

Proof. Let \(G = P_5 \times P_5\). Suppose that \(I\) be a connected decycling set of order 6 (by lemma 10.2(a)). Lemma 10.1 implies that the connected decycling set \(I\) has margin 1 so, \(I\) has at least two vertices of degree 3 or one of it has degree 2 and rest of them has degree 4. So, the pattern of removal vertices in a given graph can be readily recognize in Fig.4b. Thus, all the removal vertices are clearly independently and also their elimination does not isolate any vertex in \(G\), therefore \(G - I\) is a tree.

\[\square\]

Lemma 10.5. If \(G = P_5 \times P_8\) be grids graph and \(I\) is connected decycling set. Then \(I(1) = I(8) = \emptyset\).

Proof. Let \(G = P_5 \times P_8\) and suppose that \(I\) be a connected decycling of order 10 (by lemma 10.2(a)). Lemma 10.1 implies that, its margin is 1. It means that at least two vertices have degree 3. Suppose that \(I(1) \neq \emptyset\). Then \(N(1) = N(2) = 1\) is belongs to \(I\). It follows from lemma 10.3 that \(N(3) = 2\), so these four vertices of \(I\) lies on the leftmost three columns, in which three vertices of degree 4, while one of it has degree 3. In addition, the rest six non-adjacent vertices of degree 4 obviously belongs to the rightmost five columns. Such that as possible scenario the remaining part of graph is a forest (i.e. two components), so our supposition contradict with the main theme of paper and margin limitation.

It is given that \(I(1) = \emptyset\) and \(I(8) = \emptyset\). Then lemma 10.3 implies that \(N(2) = 2\) and \(N(3) = 1\), similarly from right to left \(N(7) = 2\) and \(N(6) = 1\). While \(N(4) = N(6) = 2\). As a result the pattern of removal independent vertices in each column of given grid is 2,1,2,2,1,2, this unique configuration readily visualized in Fig.4e. Hence, \(I(1) = \emptyset\) and \(I(8) = \emptyset\).

\[\square\]

Lemma 10.6. Every connected decycling set for \((P_5 \times P_9)\) is types of independent vertices (see Fig.4f) and their elimination leave a tree.

Proof. Suppose that \(G = P_5 \times P_9\). Let \(I\) be a CDS of order 11 (by lemma 10.2(a)). Lemma 10.1 implies that, its margin is 0. It means that there is only one vertex in \(I\) whose degree 3, while the remaining ten vertices of \(I\) has degree 4 (see Fig.4f).
It follows from lemma 10.3 that $N(1,2) = 2$, and $N(1,3) = 3$. Lemma 10.5 implies that $N(1) = \emptyset, N(2) = 2$, and $N(3) = 1$, similarly, we observe Fig.4f from right to left, where $N(9) = \emptyset$, $N(8,9) = 2$ and $N(7,9) = 3$. Hence, the vertices $v_{2,2}, v_{4,2}$ and $v_{3,3}$ are in $I$ and by symmetry $v_{2,8}, v_{4,8}$ and $v_{3,7}$ are also in $I$. Since $\tau(P_5 \times P_4) = 5$ and $\tau(P_5 \times P_5) = 6$ in this scenario $N(4) = 2$, $N(6) = 2$, and $N(5) = 1$. Lastly, the deleted vertices are independent and their removal cannot isolate any vertex in $G$, therefore $G - I$ is a tree.

![Fig.4f.](image)

**Lemma 10.7.** (i). For all $h$, $\tau(P_5 \times P_{8h+2}) = 11h + 2$.

(ii). Furthermore, if $I$ be a $\tau-$ set for $P_5 \times P_{8h+1}$ then the set $I(8k+1, 8h + 9)$ is continuity of lemma 10.6 (wher $k = 0, 1, 2 ..., h - 1$).

**Proof.** Firstly, we need to prove the stated values are upper bound, for this the set defined in (ii) will be used. Secondly, we reverse the inequalities and easily develop a proper structure of connected decycling set by induction.

It has been proved that $k = 1$, hold for base cases. Assume that they holds for $P_5 \times P_{8h+1}$ and $P_5 \times P_{8h+2}$.

**Case-1** Suppose that a graph $G = P_5 \times P_{8h+9}$ and let $I$ be the connected decycling set of $G$. So there are three possible cases.

C#1.1. *If $I(9) = \emptyset$. Then $I(1,9)$ and $I(8, 8h + 9)$, and theorem 10 and induction hypothesis holds the given structure. Thus, $I$ is independent set in $G$.*

C#1.2. *If $I(8) = \emptyset$. Then $N(1,7) \geq \tau(P_5 \times P_8) = 10$ and use induction hypothesis on $P_5 \times P_{8h+2}$, we get that $N(8, 8h + 9) \geq \tau(P_5 \times P_{8h+2})$. Hence, $|I| \geq 11h + 2$. This is clear contradiction.*

C#1.3. *$I(8) \neq \emptyset$, and $I(9) \neq \emptyset$. In such a way lemmas 10.6 implies that $N(1,8) \geq 11$ and again use induction hypothesis $(8, 8h + 9) > \tau(P_5 \times P_{8h+1}) \geq 11h$.*

This is also contradiction. Thus establish the induction step, and proof is done for $n \equiv 1(mod 8)$.

**Case-2.** Let $X = P_5 \times P_{8h+10}$ and let $I$ be a $\tau-$ set of $X$. Then we have to show that, $|I| \geq 11h + 13$. We consider three cases and its proof likely case-1.
\textbf{Case-2.1.} \( I(10) = \emptyset. \) Then \( N(1, 10) \geq 13 \) and \( N(10, 8h + 10) \geq \tau(P_5 \times P_{8h+1}) = 11 + 2, \) hence \(|I| \geq 11 + 13. \)

\textbf{Case-2.2.} \( I(9) = \emptyset. \) Then \( N(1, 9) \geq 11 \), and \( N(8, 8h + 10) \geq \tau(P_5 \times P_{8h+2}) = 11 + 2. \) Hence, again \(|I| \geq 11 + 13. \)

\textbf{Case-2.3.} \( I(9) \neq \emptyset, \) and \( I(10) \neq \emptyset. \) Then \( N(1, 9) > \tau(P_5 \times P_9) = 11. \) Lemma 10.6 implies that \( N(10, 8h + 10) \geq \tau(P_5 \times P_{8h+1}) = 11. \)

Thus the inequality satisfied for this case, hence, the lemma follows by induction. 

\textbf{Lemma 10.8.} \ Assume that \( n = 11h + r, \) with \( 1 \leq r \leq 8. \) Then \( \tau(P_5 \times P_{8h+r}) = 11h + \tau(P_5 \times P_r). \)

\textbf{Proof.} By the combination of lemma 10.7(b) and \( P_5 \times P_r, \) we have the following.

\[ \tau(P_5 \times P_{8h+r}) \leq 11h + \tau(P_5 \times P_r). \]

In order to obtain the lower bound for this grid, we reverse the inequality by using mathematical induction on \( h, \) lemma 10.7 implies that the results holds for \( r = 1 \) and 2. It is trivial for \( h = 0 \) so we can assume that \( n = 11h + r \) and consider a graph \( G = P_5 \times P_{8(h+1)+r}. \)

Let \( I \) be a connected decycling set of \( G. \) Then

\textbf{Case-i.} \( I(8) = \emptyset. \) Then \(|I| = N(1, 8) + N(8, 8h + r + 8) \geq \tau(P_5 \times P_8) + \tau(P_5 \times P_{8h+r+1}) = 10 + 11 + \tau(P_5 \times P_{r+1}) \geq 11(h + 1) + \tau(P_5 \times P_r) \) (i.e. Table.1 observations when \( r \leq 8). \)

\(|I| = 11(h + 1) + \tau(P_5 \times P_r). \)

\textbf{Case-ii.} \( I(8) \neq \emptyset. \) Then \(|I| \geq N(1, 8) + N(9, 8h + r + 8) \geq \tau(P_5 \times P_9) + \tau(P_5 \times P_{8h+r}) \geq 11 + 11h + \tau(P_5 \times P_r) \geq 11(h + 1) + \tau(P_5 \times P_r). \)

It has cleared from above two cases bounds are explicitly hold.

\textbf{Theorem 10.9.} \ Assume that \( n = 11h + r, \) with \( 0 \leq r \leq 7. \) Then \( \tau(P_5 \times P_n) = 11h + \left\lfloor \frac{3r}{2} \right\rfloor - 1. \)

\textbf{Proof.} This result follows by lemma 10.8 and lemma 10.2. For example, there are four independent subset of vertices in \( G \) (see Fig.4g).

\( I_1 = \{v_{1,5}, v_{1,13}, v_{1,21}, \ldots, v_{1,8k-3}\}; \) \( \forall k N. \)

\( I_2 = \{v_{2,2}, v_{2,4}, v_{2,6}, \ldots, v_{2,2k}\}; \) \( \forall k N. \)

\( I_3 = \{v_{3,3}, v_{3,7}, v_{3,11}, \ldots, v_{3,4k-1}\}; \) \( \forall k N. \)

\( I_4 = \{v_{5,2}, v_{5,4}, v_{5,6}, \ldots, v_{5,2k}\}; \) \( \forall k N. \)
\[ I = I_1 \cup I_2 \cup I_3 \cup I_4. \]

Now we may see a complete symmetry in Fig.4g, where all the removal vertices are exactly independent and their deletion cannot isolate any vertex in \( G \), therefore \( G - I \) is a tree.

Fig.4g.

Theorem 11. If \( m = 6 \) and \( n \geq 4 \), then \( \tau(P_6 \times P_n) = \left\lfloor \frac{5n}{3} \right\rfloor \).

In the proof of theorem 11, we will use the following two lemmas.

Lemma 11.1. \( \tau(P_6 \times P_6) = 10 \).

Proof. Suppose that \( G = P_6 \times P_6 \) and let \( I \) be a connected decycling set of order 9. The graph contains nine disjoint 4-cycles, so, obviously \( \tau(P_6 \times P_6) \geq 10 \).

Corollary 3, implies that its margin is 1, it means that \( I \) has more than 1 boundary vertices. In addition, lemma 5, stated that at least one vertex of degree 3 or 2 lies in the first two columns and the same must hold for last two columns. So, the rest of three independent vertices in \( G \) of degree 4, does not sufficient to remove all cycles in a given grid, therefore we need one more vertex of degree 3 or 4 in middle columns to make it cycle free (see Fig.5a). Thus, the connected decycling set of size is 10.

Lemma 11.2. \( \tau(P_6 \times P_9) = 15 \).

Proof. Suppose that \( G = P_6 \times P_9 \) has twelve disjoint 4-cycles. Theorem 4, implies that \( \tau(P_6 \times P_6) \geq 14 \). Let \( I \) be a connected decycling set of order 14 and it follows from corollary 2, that \( m(I) \geq 1 \). Then at least two independent vertices of degree 3. Similarly, lemma 11.1, those twelve independent interior independent vertices of \( G \) cannot complete the connected decycling; therefore, we need to remove three more vertex from middle columns. From above discussion, it is conformed that \( N(1) = 1, N(9) = 1 \), and \( N(2,8) = 13 \). The arrangement of independent vertices in \( G \) are 1,2,2,2,1,2,2,2,1 can be readily recognized in Fig.5b, and its elimination also does not isolate any vertex in \( G \), Thus, the connected set size is 15. This is the complete of lemma 11.2.
Proof of Theorem 11. The lemma 11.1, lemma 11.2, with lemmas 3, shows that \( \tau(P_6 \times P_n) \leq \left\lfloor \frac{5n}{3} \right\rfloor \). In order to reverse the inequality, we use induction on \( h \). We assume that \( n = 6h \). Then \( \tau(P_6 \times P_{6h}) \geq 10h \), it is true for \( h = 1 \) (by lemma 11.1) since \( \tau(P_6 \times P_{6h+6}) = \tau(P_6 \times P_{6h}) + \tau(P_6 \times P_6) \geq 10 + 10h \). In a similar fashion lemma 10.2, implies that

\[
\tau(P_6 \times P_{6h+9}) = \tau(P_6 \times P_9) + \tau(P_6 \times P_{6k}) \geq 15 + 10h = 5(2 + 3h).
\]

Hence, there are five sub-vertex set in , it may be readily recognized in Fig.5b.

\[
I_1 = \{v_{1,1}\}.
\]

\[
I_2 = \{v_{2,3}, v_{2,5} \ldots v_{2,2k+1}\}; \quad \forall \ k \in N.
\]

\[
I_3 = \{v_{2,2}, v_{2,4} \ldots v_{2,2k}\}; \quad \forall \ k \in N.
\]

\[
I_4 = \{v_{5,2}, v_{5,4} \ldots v_{5,2k}\}; \quad \forall \ k \in N.
\]

\[
I_5 = \{v_{6,3}, v_{6,5} \ldots v_{6,4k-1}\}; \quad \forall \ k \in N.
\]

\[
I = I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5
\]

The connected decycling of maximum size is \( 5(3h + 2) \), and at the same time both basic conditions are satisfied, therefore \( G - I \) is a tree. ■

Theorem 12. If \( m = 7 \) and \( n \geq 4 \), then \( \tau(P_7 \times P_n) = 2n - 1 \).

Proof. It is easy to find the connected decycling number of given grids graph because it follows from theorem 4, that \( \tau(P_7 \times P_n) \geq \left\lfloor \frac{6n-4}{3} \right\rfloor = 2n - 1 \). Lemma 3, implies that \( I \) has 0 margin, since at least one of its vertex has degree 3 and rest of them degree 4. So, the degree 3 vertex lies in column one, while each column has exactly two removal vertices (i.e. 1, 2,2, 2...) and the symmetry of eliminated vertices are depicted in Fig.6.

\[
I_1 = \{v_{2,2}, v_{2,4} \ldots v_{2,2k}\}; \quad \forall \ k \in N. \quad I_2 = \{v_{3,3}, v_{3,5} \ldots v_{3,2k+1}\}; \quad \forall \ k \in N.
\]
\[ I_3 = \{ v_{4,1} \}. \]

\[ I_4 = \{ v_{5,3}, v_{5,5} \ldots v_{5,2k+1} \} \forall \ k \in N \]

\[ I_5 = \{ v_{6,2}, v_{6,4} \ldots v_{6,2k} \}, \forall \ k \in N. \]

\[ I = I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5. \] Therefore, \( G - I \) is a tree.

For the general case, we introduce another vertex set construction in \( P_m \times P_n \) which enable us to study the connected decycling set in dense graph \( G \), so our new construction allows using CDS in small grids build a CDS in large grids graph.

In order to construct a new CDS, we insert a new edge-vertex and cell-vertex in the middle of each edge and cell of \( P_m \times P_n \) respectively, and joining each new cell-vertex to the new edge-vertices on perimeter of cell. In such a way, we obtained a new large grid and it designated by \((P_m \times P_n)_{Ex}\), actually this grid is a copy of \( P_{2m-1} \times P_{2n-1} \). This type of construction covers a good enough part of total grid. Moreover, we ignore all those finite classes of \( P_m \times P_n \) where theorem 13 does not apply.

**Theorem 13.** If \( m = n = 2^z + 1 \), where \( z \geq 2 \). Then \( \tau(P_m \times P_n)_{Ex} = \tau(P_{2m-1} \times P_{2n-1}) \).

**Proof.** Let \( G = P_m \times P_n \) be a grids graph, and \((P_m \times P_n)_{Ex}\) is its expanded graph. Suppose that \( I \) and \( I_{Ex} \) be its connected decycling sets respectively. Actually, \( I_{Ex} \) is the union of \( I \) and newly inserted cell-vertices. This useful construction can be readily recognized in Fig.7a, and Fig.7b where the black labeled vertices in Fig.7b represents the connected decycling set corresponding to the black vertices in Fig.7a, and the green vertices in Fig.7b signifies the newly inserted cell-vertices. If we eliminate all black and green vertices \( I_{Ex} \) from \((P_m \times P_n)_{Ex}\), then the rest graph become a maximum tree , that is, \( (P_m \times P_n)_{Ex} - I_{Ex} = T \).

Such that, the constructed graph shows that the lower bound \( \tau(P_{2m-1} \times P_{2n-1}) \) on the size of connected decycling set \( I_{Ex} \) of \((P_m \times P_n)_{Ex}\) is equal to the lower bound \( \tau(P_m \times P_n) \) on the size of the connected decycling set \( I \) of \((P_m \times P_n)\) plus cycle rank of \( G \). That is
\[ \tau(P_m \times P_n)_{Ex} = \tau(P_m \times P_n) + \beta(P_m \times P_n). \]
\[ = \left\lfloor \frac{(\beta(G)+1)}{3} \right\rfloor + \beta(G). \] 
(i.e. by Corollary 2)
\[ = \left\lfloor \frac{4(\beta(G))}{3} + 1 \right\rfloor. \]
\[ = \left\lfloor \frac{4(mn - m - n + 1) + 1}{3} \right\rfloor \]
\[ = \left\lfloor \frac{4mn - 4m - 4n + 5}{3} \right\rfloor \]

The lower bound \( \tau(P_{2m-1} \times P_{2n-1}) \) on the size of connected decycling set of \( G \) is \( \left\lfloor \frac{4mn - 4m - 4n + 5}{3} \right\rfloor \). Hence, \( \tau(P_m \times P_n)_{Ex} = \tau(P_{2m-1} \times P_{2n-1}) \).

**Remarks.** If \( I \) is connected decycling set of \( P_m \times P_n \), then \( I_{Ex} \) is CDS of \( (P_m \times P_n)_{Ex} \), of size \( \tau(P_m \times P_n)_{Ex} \). Specially, the ‘distance’ a connected decycling set is away from the lower bounds \( \tau(P_m \times P_n) \) in \( G \) is very carefully preserved by construction in \( P_{2m-1} \times P_{2n-1} \). That is \( |I_{Ex}| - \tau(P_{2m-1} \times P_{2n-1}) = |I| - \tau(P_m \times P_n) \).

So, we almost succeed to find the connected decycling sets in grids \( P_{2^z+1} \times P_{2^z+1} \) for \( z \geq 2 \)

**Conclusion**

In this paper, we have successfully find the connected decycling number of \( (P_m \times P_n) \), which, in some sense and this important analysis put a great impact in combinatorial circuit designing and operating tasks in computer science. Of course, this problem still demands further investigation for the rest cases of 2-dimentional grids graph.
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